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# REGULARITY OF POWERS OF SOME IDEALS

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## INTRODUCTION

Let $A = K[x_1, \cdots, x_d]$ be a polynomial ring over a field $K$ and $\mathfrak{m} = (x_1, \cdots, x_d)$. We regard $A$ as a graded object with some positive degree $deg(x_i) = w_i$ for $i = 1, \ldots, d$. Let $I$ be a graded ideal of $A$. In this note, we consider the regulaty $\operatorname{reg}(I^n)$ for all $n \geq 0$. For a graded $A$-module $M$, $\operatorname{reg}(M)$ is define to be the following,

$$\operatorname{reg}(M) = max\{reg_i(M) \mid i \geq 0\}$$

where $\operatorname{reg}_i(M) = max\{a \mid [Tor_A^i(K, M)]_{a-i} \neq 0\}$. In other word, $\operatorname{reg}(M)$ is a maximal degree shits in a graded minimal $A$-free resolution of $M$.

In thier paper[1], Cutkosky-Herzog-Trung showed the following theorem.

**Theorem.** *Let $I$ be a graded ideal of $A$ and $i \geq 0$. Then there exist integrers $c_i(I)$ and $d_i(I)$ such that*

$$\operatorname{reg}_i(I^n) = c_i(I)n + d_i(I)$$

*for every sufficiently large $n$. Furthermore, $reg(I^n)$ is also linear and a leading coefficient coincides with $c_0(I)$.*

It is natural to ask the following.

**Question.**

(1) To describe the function $\operatorname{reg}(I^n)$, precisely.
(2) What is the smallest number $\operatorname{reg}(I^n)$ to be linear.

We put $s = min\{t \mid \operatorname{reg}(I^n)$ is linear for all $n \geq t\}$. It is easy to see that the constant term of $\operatorname{reg}(I^n)$ is $\operatorname{reg}(I^s) - c_0(I)s$. Thus it is enough to decide $c_0(I)$ for describing $\operatorname{reg}(I^n)$.

## 1. ABOUT $c_0(I)$

$c_0(I)$ is closely related to a reduction of $I$. We first define the following numbers arising from reduction ideals.

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**Definition 1.1.** *We set*

$$rdeg(I) = min\{\mathrm{reg}_0(J) \mid J \subset I \text{ is a graded reduction ideal}\}.$$

An element $a \in I$ said to be reduced modulo $\mathfrak{m}I$, if each homogeneous components of $a$ is nonzero in $I/\mathfrak{m}I$. Also, a sequence $a_1, \cdots, a_t \in I$ is called reduced modulo $\mathfrak{m}I$, if every $a_i$ is reduced modulo $\mathfrak{m}I$.

Now, we give an answer of Question (1) as follows.

**Proposition 1.2.** *Let $I$ be a graded ideal of $A$. Assume that $I$ has a minimal reduction. Then $c_0(I) = rdeg(I)$. More precisely, if $a_1, \cdots, a_l \in I$ is a minimal reduction which is reduced modulo $\mathfrak{m}I$, then $c_0(I) = rdeg(I) = deg(a_1, \cdots, a_l)$, where $deg(a_1, \cdots, a_l) = max\{\deg(a_i) \mid i = 1, \cdots, l\}$.*

**Proof.** Let $a_1, \cdots, a_l \in I$ be a minimal reduction of $I$ and $c = deg(a_1, \cdots, a_l)$. A reduction property does not depend on the difference of elements of $\mathfrak{m}I$. Hence, we may assume that $a_1, \cdots, a_l$ is reduced modulo $\mathfrak{m}I$.

Let $J'$ be a graded ideal generated by all homogeneous components of $a_1, \cdots, a_l$. (Note that $J'$ depends on the choice of minimal generaters of $J$).

Then $J'$ becomes a reduction of $I$. Indeed, we have the following inculsions for all $n >> 0$

$$I^n = JI^{n-1} \subset J'I^{n-1} \subset I^n.$$

This shows that $rdeg(I) \leq c$.

For a graded reduction $J \subset I$, if $I^{n+r} = J^n I^r$ for $n \geq 0$, then

$$\mathrm{reg}_0(I^{n+r}) \leq \mathrm{reg}_0(J^n) + \mathrm{reg}_0(I^r) \leq \mathrm{reg}_0(J)n + \mathrm{reg}_0(I^r)$$

for all $n \leq 0$. Hence we have

$$c_0(I) = \lim_{n \to \infty} \frac{reg_0(I^n)}{n} \leq reg_0(J).$$

This implies that $c_0(I) \leq rdeg(I)$.

Finally, we will show that $c \leq c_0(I)$. We may assume that $deg(a_1) = c$ and denote by $b$ a homogeneous component of $a_1$ in degree $c$. Since $a_1, \cdots, a_l$ is analitically independent, $b^n$ ( a head term of $a_1^n$) is nonzero in $I^n/\mathfrak{m}I^n$ for all $n > 0$. In other words, $[I^n/\mathfrak{m}I^n]_{cn} \neq 0$. Thus

$$cn \leq max\{t \mid [I^n/\mathfrak{m}I^n]_t \neq 0\} = \mathrm{reg}_0(I^n).$$

This shows that $c \leq c_0(I)$ in the same way as above. Hence we have $c_0(I) = rdeg(I) = c$.

At this moment, we don't have enough tool solving Question (2). In the next section, we give a trivial answer for the simple situation.

## 2. REGULARITY FOR D-SEQUENCES

In this section, we prove the following.

**Theorem 2.1.** *Let $I \subset A$ be a ideal generated by monomial d-sequence. Then* $\mathrm{reg}(I^n) = reg_0(I)n + (reg(I) - reg_0(I))$.

Recall that a sequence $a_1, \cdots, a_r$ of elements of $A$ is a *d-sequence* (cf. [3]), if it generates $(a_1, \ldots, a_r)$ minimally and satisfies the following condition

$$(a_1, \cdots, a_i) : a_{i+1}a_j = (a_1, \cdots, a_i) : a_j$$

for every $1 \leq i < j \leq r$.

By results of [4], we can construct a free resolution of the Rees algebra of $(a_1, \cdots, a_r)$. Such a resolution contains $A$-free resolutions of $I^n$. In our case, these $A$-free resolutions are reduced to be minimal. Thus we can compute $reg(I^n)$ for a monomal d-sequence.

In the following, we give a construction of resolutions.

Let $a_1, \cdots, a_r$ be a d-sequence and $I = (a_1, \cdots, a_r) \subset A$. We set $S = A[T_1, \cdots, T_r]$ and $deg(T_i) = 1$ for $i = 1, \cdots, r$. (At this moment, we don't consider the grading on $A$. In fact, the following argument is possible for any ring. Thus we regard $deg(a) = 0$ for $a \in A$ in the grading on $S$.)

We put $\mathcal{Z}_i(I, S) = Z_i(I) \otimes_A S(-i)$ for $i = 0, \cdots, r$ where $Z_\bullet(I)$ is a cycle of a Koszul complex of $I$. Then the Koszul complex $K_\bullet(T_1, \cdots, T_r; S)$ induces

$$0 \to \mathcal{Z}_r(I, S) \to \cdots \to \mathcal{Z}_2(I, S) \to \mathcal{Z}_1(I, S) \to \mathcal{Z}_0(I, S) \to 0$$

,so call $\mathcal{Z}$-complex. By [4], if $I$ is generated by a d-sequence, then $\mathcal{Z}_\bullet(I, S)$ is acyclic with 0th homology isomorphic to the Rees algebra $R(I)$.

Let $P_\bullet^{(i)}(I)$ be a $A$-free resolution of $Z_i(I)$ $(i = 0, \cdots, r)$ and $P_{i,\bullet}(I, S) = P_\bullet^{(i)}(I) \otimes S(-i)$. Then the differentials $\mathcal{Z}_i(I, S) \to \mathcal{Z}_{\imath-1}(I, S)$ lifts to a chain map $\varphi : P_{i,\bullet}(I, S) \to P_{i-1,\bullet}(I, S)$ and $P_{\bullet,\bullet}(I, S)$ becomes a $S$-double complex.

By the stadard arguments of a spectral sequence and a cyclicity of $\mathcal{Z}_\bullet(I, S)$, the associated total complex $Tot(P_{\bullet,\bullet}(I, S))$ gives a $S$-free resolution of the Rees algebra $R(I)$.

In this case, $Tot(P_{\bullet,\bullet}(I, S))$ is not only acyclic, but also it has some information abou the differential $\varphi$. If we put $I' = (a_1, \cdots, a_{r-1})$, then

$$0 \to Z_\bullet(I') \to Z_\bullet(I) \to Z_\bullet(I')[-1] \to 0$$

is exact.

Now, we consider the monomial case. Assume that $a1, \cdots, a_r$ is a monomial d-sequence.

For $F \subset [r]$ and $1 \leq i \leq r$, we set

$$a_F^{(i)} = \begin{cases} LCM(\prod_{j \in G} a_j \mid G \subset F, \, {}^\#G = i), & ({}^\#F \geq i) \\ 0, & ({}^\#F < i). \end{cases}$$

Then we can choose that $P^{(i)}_{\bullet}(I) = \bigoplus_{F\in[n],|F|>i} Ae^{(i)}_I$ with a differential $\partial$

$$\partial(e^{(i)}_F) = \sum_{j\in F} \sigma(j,F)\frac{a^{(i)}_F}{a^{(i)}_{F\setminus\{j\}}} e^{(i)}_{F\setminus\{j\}}.$$

((2.3) in [5]) Furthermore, the lifting $\varphi : P_{i,\bullet}(I,S) \to P_{i,\bullet}(I,S)$ is give by

$$d(e^{(i)}_F \otimes 1) = (-1)^{|F|-i} \sum_{j\in F} \sigma(j,F)\frac{a^{(i)}_F}{a^{(i-1)}_{F\setminus\{j\}} a_j} T_j e^{(i-1)}_{F\setminus\{j\}} \otimes 1.$$

Then there is a exact sequence

$$0 \to P_{\bullet,\bullet}(I',S) \to P_{\bullet,\bullet}(I,S) \to P_{\bullet,\bullet}(I',S)[-1,0] \to 0$$

of double complexes. Then, by induction on $r$, the following is also exact

$$C = \cdots \to P_{3,\bullet}(I,S) \to P_{2,\bullet}(I,S) \to \varphi(P_{2,\bullet}(I,S)) \to 0.$$

Finally, we have the exact sequence $Tot(P_{\bullet,\bullet}(I,S)/C \cong P_{1,\bullet}(I,S)/\varphi(P_{2,\bullet}(I,S))$ and this is actually $A$-free . After the small Gröbner basis computation, this resolution is wriiten in the following form.

**Proposition 2.2.** *Let $\Sigma = \{(F,\alpha) \mid F \subset [r], \alpha \in \mathbb{N}, max(F) \geq max(\alpha)\}$. Here we denote $max(F)$ is a maximal number in $F$ and $max(\alpha) = max(supp(\alpha))$. Then $I^n$ has a free resolution $P_\bullet$ of the form*

$$P_i = \bigoplus_{(F,\alpha)\in\Sigma} Ae^{(i)}_F \otimes T^\alpha.$$

*Furthermore, if $a_r$ does not divide $LCM(a_1,\cdots,a_{r-1})$, then the above $A$-free resolution is minimal.*

At last, it is easy to compute degrees of $e^{(i)}_F \otimes T^\alpha$, and then we have

$$reg(I^n) = reg_0(I)n + (reg(I) - reg_0(I))$$

for all $n > 0$.

## References


[1] D. Cutkosky - J. Herzog - N.V. Trung. Asymptotic behaviour of the Castelnuovo - Mumford regularity. Preprint 1997.

[2] D. Eisenbud. *Commutative Algebra with a view to algebraic geometry.* Springer 1995.

[3] C. Huneke. The theory of $d$-sequences and powers of ideals. Adv. in Math. **46** (1982), 249-279.

[4] J. Herzog -A. Simis - V. Vasconcelos. Approximation complexes and blowing-up rings. J. of Alg. **74(2)**, (1982), 466-493.

[5] J. Herzog -Y. Kamoi. Taylorcomplexes for Koszul boundaries, manuscripta math. **96**, (1998), 133-147.

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